arXiv:math/9808122v1 [math.RA] 28 Aug 1998

PRIMITIVE IDEALS IN HOPF ALGEBRA EXTENSIONS

MARK C. WILSON

Abstract. Let H be a finite-dimensional Hopf algebra. We study the behaviour of primitive and maximal ideals in certain types of ring extensions determined by H. The main focus is on the class of faithfully flat Galois extensions, which includes includes smash and crossed products. It is shown how analogous results can be obtained for the larger class of extensions possessing a total integral, which includes extensions $A^H \subseteq A$ when H is semisimple. We use Passman's "primitivity machine" to reduce the whole theory of Krull relations for prime ideals to the case of primitive ideals. The concept of strongly semiprimitive Hopf algebra is introduced and investigated. Several examples and open problems are discussed.

1. Introduction

A series of papers by Schneider [Sch90a, Sch90b, Sch92] has made clear the importance of the class of faithfully flat Hopf Galois extensions. As well as including smash and crossed products, such extensions arise in the study of algebraic groups [Sch90a]. A large class of interesting examples is provided by the following situations. Let A be a Hopf algebra with a normal Hopf subalgebra R such that either (i) A is pointed or the coradical of A is cocommutative or (ii) R is central and of finite index, and R is a noetherian ring. Then A is a faithfully flat H-Galois extension of R, where $H = A/AR^+$ and R^+ denotes the augmentation ideal of R. For more examples, see [MSch] and [Sch90b]; the former contains an explicit quantum group example.

The abstract viewpoint of faithfully flat Hopf Galois extensions makes it possible to give unified proofs of seemingly disparate results. There are several technical advantages to working within the larger class of extensions, even if one is only interested in crossed products. One of these is transitivity, as defined in Section 2. Another is stability under various natural operations. Thus even for connected Hopf algebras, for which every faithfully flat Hopf Galois extension is in fact a crossed product [Bel], there is value in this approach.

In [MSch], Montgomery and Schneider made a thorough examination of the behaviour of prime ideals in faithfully flat Hopf Galois extensions, with particular attention to the Krull relations. The present paper contains an analogous development for primitive ideals, and discusses the relation between the two theories. In addition we rework the presentation of some fundamental topics in [MSch], and give a more leisurely treatment of others, in order to allow future work to proceed more

Date: February 1, 2008.
1991 *Mathematics Subject Classification.* Primary 16W30. Secondary 16S40.
Key words and phrases. faithfully flat, Galois extension, maximal ideal.

smoothly. We hope to stimulate further work on this subject, and so several conjectures, most of them not new, are explicitly stated.

The outline of the paper is as follows. In Section 2 we recall the main definitions and general results from previous papers. Much of this material is contained in [MSch]. The chief novelty lies in our presentation, including the systematic use of commutative diagrams to make explicit various reductions which simplify much of the later work. Results and notation from this section are used heavily throughout the paper.

Section 3 concerns Krull relations in H-extensions. We give a more leisurely treatment of some topics covered in [MSch]. Furthermore we introduce what we call internal Krull relations, which seem to be technically useful.

If all the Krull relations hold, then many properties of maximal ideals of R yield corresponding results for A, and vice versa. In view of the difficulty of verifying all Krull relations, however, it may be useful to consider maximal ideals separately. Section 4 deals in some detail with basic questions about maximal and H-maximal ideals.

Section 5 deals with modules and annihilators, in preparation for a discussion of primitive ideals. We make explicit certain module-theoretic properties (the finite and semisimple induction and restriction properties) which have proved crucial in group and enveloping algebra arguments. We treat primitive and H-primitive ideals in Section 6. The aim (largely fulfilled) is to prove analogues for Prim of all results about Spec from [MSch]. In addition, we use Passman's "primitivity machine" to relate results on the two spectra, and show thereby that the theory may be founded on Prim.

A brief Section 7 discusses the relaxation of our standing hypotheses to the larger class of H-extensions with a total integral. This is equivalent to studying H-module algebras with surjective trace map, a condition which is always satisfied when H is semisimple. Because this procedure has been done many times before, and is extensively discussed in [MSch], we omit many details. Section 8 discusses various obviously defined H-radicals and their relation to other properties. In particular we focus on a property of semisimple H, strong semiprimitivity, which relates to the module properties mentioned above.

In Section 9, we discuss several classes of Hopf algebras, and summarize which properties discussed in this paper they are known to satisfy. One new result here, answering a question from [MSch], is that every pointed H satisfies the Krull relation t-LO. We do not pursue specific examples of Hopf algebras or extensions in detail, as this paper's goal is foundational. However, there should be several quantum group applications of the content of this paper. We conclude with some open problems in Section 10.

Notation and conventions. Let F be a field and let H be a Hopf algebra over F. We shall be concerned with H-extensions, that is, extensions $R \subseteq A$ of F-algebras such that A is an H-comodule algebra via $\rho : A \to A \otimes H$ and $R = A^{\operatorname{co} H}$ is its algebra of coinvariants. If H is finite-dimensional, then this is equivalent to A being an H^*-module and R the algebra of invariants for H^*. The notation $\mathcal{E} = (R, A)$ will be used for H-extensions.

A general reference for all unproved assertions about Hopf algebras which cannot be found in [MSch] is the book [Mon93].

2. Faithfully flat Galois extensions

In this section we consider the fundamentals of faithfully flat Galois extensions. The results in this section allow for great simplification of proofs later on, and will be used extensively.

The H-extension $\mathcal{E} = (R, A)$ is said to be *Galois* if the map $A \otimes_R A \to A \otimes H$ given by $x \otimes y \mapsto x\rho(y)$ is bijective. If in addition A is faithfully flat as a left R-module, we say that $\mathcal{E}$ is *faithfully flat H-Galois*. If H is finite-dimensional and we do not wish to specify H, we shall say that $\mathcal{E}$ is a faithfully flat finite Galois extension.

To say that H has a given property (normally associated with extensions) will mean that all faithfully flat H-Galois extensions have the property.

Suppose now that (R, A) is faithfully flat H-Galois. A fundamental fact [Sch90a] is that there is an equivalence between the category of left R-modules and the category of left A-modules which are also right H-comodules, given by $V \mapsto AV$ and $W \mapsto W^{\mathrm{co}\, H}$. We shall use a consequence of this heavily in the special case of ideals.

An ideal I of R is said to be *H-stable* if $IA = AI$. If the antipode of H is bijective, and the extension is a crossed product, then the usual definition of H-stable implies the above property. For general H this is no longer the case, however.

The set of all H-stable ideals of R forms a sublattice $\mathcal{I}_H(R)$ of the lattice $\mathcal{I}(R)$ of all ideals of R.

2.1. Definition. For each ideal I of R, the *H-core* $(I : H)$ is the largest H-stable ideal of R contained in I.

The H-core is well-defined since the sum of H-stable ideals is H-stable. Taking the H-core gives a map $\mathcal{I}(R) \to \mathcal{I}_H(R)$ which is a lattice epimorphism. The set-theoretic kernel of this map is a partition of $\mathcal{I}(R)$ whose associated equivalence relation we denote by $\sim_H$. Explicitly, $Q \sim_H Q'$ if and only if $(Q : H) = (Q' : H)$. Note that each equivalence class has a unique H-stable member, the common H-core of all elements in the class.

The category equivalence above, when restricted to ideals, leads to a fundamental correspondence between H-stable ideals of R and ideals of A which are H-subcomodules. When H is finite-dimensional, this latter set coincides with the the set of H^*-stable ideals of A. In any case, we shall denote it by $\mathcal{I}_{H^*}(A)$. The correspondence is given by expansion $(I \mapsto IA)$ and contraction $(J \mapsto J \cap R)$. The following result from [MSch] gives the precise information we require.

2.2. Proposition. *Let (R, A) be a faithfully flat H-Galois extension. Then expansion and contraction yield maps $\Phi : \mathcal{I}_H(R) \to \mathcal{I}_{H^*}(A)$ and $\Psi : \mathcal{I}_{H^*}(A) \to \mathcal{I}_H(R)$ which are mutually inverse lattice isomorphisms.* □

Stability properties. We now discuss the stability of faithfully flat Galois extensions under certain operations. In the following, a property of ring extensions is identified with the class of all extensions possessing the property.

2.3. Definition. Let $\mathcal{P}$ be a property of ring extensions. Say that $\mathcal{P}$ is *transitive* if whenever $R \subset B \subset A$ and both (R, B) and (B, A) have the property then (R, A) has the property. If $\mathcal{E}$ is an extension of F-algebras, and E is a subfield or extension field of F, then $\mathcal{P}$ *holds over E* if and only

if the extension formed from $\mathcal{E}$ by (respectively) restriction or extension of scalars to E satisfies $\mathcal{P}$. If $\mathcal{P}$ holds over all subfields and extension fields, then we say that $\mathcal{P}$ is *field-independent*.

In the next result, if whenever all factors of a subnormal series for H possess a given property $\mathcal{P}$, then H has $\mathcal{P}$, we say that $\mathcal{P}$ *ascends via subnormal series*.

2.4. Proposition. *Let (R, A) be a faithfully flat H-Galois extension, where H is finite-dimensional. If K is a normal Hopf subalgebra of H with quotient $\overline{H}$, then there is an intermediate subalgebra B such that (R, B) is faithfully flat K-Galois and (B, A) is faithfully flat $\overline{H}$-Galois. Consequently, transitive properties ascend via subnormal series.* □

2.5. Proposition. *The property of being faithfully flat finite Galois holds over extension fields and subfields, and respects quotients by stable ideals. More precisely, let (R, A) be a faithfully flat H-Galois extension, let I be an H-stable ideal of R, let E_1 be a subfield of F and let E_2 be an extension field of F. Then*

(i) *(R', A') is faithfully flat H'-Galois, where $'$ denotes extension of the ground field to E_2*
(ii) *(R, A) is faithfully flat H-Galois when R, A, H are considered as E_1-algebras*
(iii) *$(R/I, A/AI)$ is faithfully flat H-Galois, where R/I is identified with its image in A/AI under the canonical map $A \to A/AI$.*

□

Duality. If H is finite-dimensional, we shall let $\mathcal{E}^*$ denote the *dual extension* $(A, A\#H^*)$. Note that H acts on $A\#H^*$ by acting trivially on A and with the usual action on H^*. Thus we can form $A\#H^*\#H$ which is Morita-equivalent to A.

2.6. Definition. Let $\mathcal{P}$ be a property of faithfully flat finite Hopf Galois extensions. The *dual* $\mathcal{P}^*$ of $\mathcal{P}$ is the class of all duals of extensions in $\mathcal{P}$.

In terms of our notational conventions, H has $\mathcal{P}$ if and only if H^* has $\mathcal{P}^*$.

2.7. Proposition. *Let $\mathcal{P}$ be a property of finite-dimensional Hopf algebras. If $\mathcal{P}$ ascends via subnormal series then so does $\mathcal{P}^*$. If $\mathcal{P}$ respects field extension/restriction then so does P^*.*

Proof. If $K \to H \to \overline{H}$ is an exact sequence (that is, K is a normal Hopf subalgebra of H and $\overline{H} = H/HK^+$), then by dualizing we obtain $\overline{H}^* \to H^* \to K^*$ which is also exact, proving the first part. The second follows from the fact that for every field E with $E \supset F$, $(H \otimes_F E)^*$ is naturally isomorphic to $H^* \otimes_F E$ as a Hopf algebra over E. □

Equivalences. In this subsection H will be assumed to be finite-dimensional. A basic fact is that R and $A\#H^*$ are Morita equivalent. In the following key result from [MSch], Φ_1, Φ_2 denote the map Φ from 2.2 in the extensions (R, A) and $(A, A\#H^*)$, respectively.

2.8. Theorem. *Let (R, A) be a faithfully flat H-Galois extension, with H finite-dimensional. Then the Morita equivalence between R and $A\#H^*$ induces a lattice isomorphism $f : I \mapsto I^\dagger$ from $\mathcal{I}(R)$ to $\mathcal{I}(A\#H^*)$. This map preserves products, and $(I^\dagger : H) = \Phi_2 \circ \Phi_1((I : H))$.* □

Thus given the chain of extensions $R \subset A \subset A\#H^*$ we may freely expand and contract stable ideals in the natural way.

We wish to develop this correspondence further, into a correspondence between extensions.

2.9. Theorem. *The following diagram commutes. All vertical maps are induced by the Morita equivalence as in 2.8. The middle horizontal and diagonal maps come from 2.2.*

$$\begin{tikzcd}
\mathcal{I}(R) \arrow[r, "(:H)"'] \arrow[d, leftrightarrow] & \mathcal{I}_H(R) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{I}_{H^*}(A) \arrow[d, leftrightarrow] & \mathcal{I}(A) \arrow[l, "(:H^*)"] \arrow[d, leftrightarrow] \\
\mathcal{I}(A\#H^*) \arrow[r, "(:H)"] & \mathcal{I}_H(A\#H^*) \arrow[r, leftrightarrow] \arrow[ur, leftrightarrow] & \mathcal{I}_{H^*}(A\#H^*\#H) & \mathcal{I}(A\#H^*\#H) \arrow[l, "(:H^*)"']
\end{tikzcd}$$

Proof. This all follows from 2.8. First note the two central triangles commute since $(I : H)^\dagger = \Phi_2 \circ \Phi_1((I : H))$. This then shows that $(I : H)^\dagger = (I : H)(A\#H^*)$ and so the left square commutes, as does the right. $\square$

It is clear from 2.9 that as far as ideals are concerned, the extensions (R, A) and $(A\#H^*, A\#H^*\#H)$ are equivalent in a strong sense. In order to verify many results on ideals in faithfully flat finite Galois extensions, it will suffice to consider smash products.

We also wish to deal with field extension and contraction in a similar way. Let E be an extension field of F, and adopt the notation of 2.5.

2.10. Theorem. *The following diagram commutes. The maps marked c are contraction, and the other vertical maps are induced by field extension and restriction. The middle horizontal maps come from 2.2.*

$$\begin{tikzcd}
\mathcal{I}(R') \arrow[r, "(:H')"] \arrow[d, "c"'] & \mathcal{I}_{H'}(R') \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{I}_{H'^*}(A') \arrow[d, leftrightarrow] & \mathcal{I}(A') \arrow[l, "(:H'^*)"'] \arrow[d, "c"] \\
\mathcal{I}(R) \arrow[r, "(:H)"'] & \mathcal{I}_H(R) \arrow[r, leftrightarrow] & \mathcal{I}_{H^*}(A) & \mathcal{I}(A) \arrow[l, "(:H^*)"]
\end{tikzcd}$$

Proof. By the previous result, we may assume that $A = R\#H$ and $A' = R\#H'$. Then it is clear from the definition of the action of H' on R' that for each ideal I of R', $(I : H') \cap R = ((I \cap R) : H)$, so the outer squares commute. Also, the middle square commutes because $(H')^* = (H^*)'$. $\square$

For modules, the situation is similar to the above. The map $g : R$–Mod $\to R'$–Mod which takes M to $M \otimes_F E$ has the property that the lattice of submodules of $g(M)$ is isomorphic to the lattice of submodules of M. This property is also shared by the map induced by Morita equivalence. Both of these maps also commute with induction and restriction. We summarize the latter fact below.

2.11. Theorem. *The following diagrams commute. The vertical maps are induced by the Morita equivalence.*

$$\begin{tikzcd} R\text{–Mod} \arrow[r, "\mathrm{ind}"] \arrow[d, leftrightarrow] & A\text{–Mod} \arrow[d, leftrightarrow] \\ A\#H^*\text{–Mod} \arrow[r, "\mathrm{ind}"] & A\#H^*\#H\text{–Mod} \end{tikzcd} \qquad \begin{tikzcd} R\text{–Mod} \arrow[d, leftrightarrow] & A\text{–Mod} \arrow[l, "\mathrm{res}"'] \arrow[d, leftrightarrow] \\ A\#H^*\text{–Mod} & A\#H^*\#H\text{–Mod} \arrow[l, "\mathrm{res}"'] \end{tikzcd}$$

The following diagrams commute. The vertical maps are induced by field extension.

$$\begin{tikzcd} R\text{–Mod} \arrow[r, "\mathrm{ind}"] \arrow[d] & A\text{–Mod} \arrow[d] \\ R'\text{–Mod} \arrow[r, "\mathrm{ind}"] & A'\text{–Mod} \end{tikzcd} \qquad \begin{tikzcd} R\text{–Mod} \arrow[d] & A\text{–Mod} \arrow[l, "\mathrm{res}"'] \arrow[d] \\ R'\text{–Mod} & A'\text{–Mod} \arrow[l, "\mathrm{res}"'] \end{tikzcd}$$

The following diagrams commute. The vertical maps are induced by field restriction.

$$\begin{tikzcd} R\text{–Mod} \arrow[r, "\mathrm{ind}"] & A\text{–Mod} \\ R'\text{–Mod} \arrow[r, "\mathrm{ind}"] \arrow[u] & A'\text{–Mod} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} R\text{–Mod} & A\text{–Mod} \arrow[l, "\mathrm{res}"'] \\ R'\text{–Mod} \arrow[u] & A'\text{–Mod} \arrow[l, "\mathrm{res}"'] \arrow[u] \end{tikzcd}$$

Proof. The category equivalence between R-Mod and $A\#H^*$-Mod is given by $V \mapsto A \otimes_R V$. For any Galois extension and any $A\#H^*$-module M, there is a natural isomorphism $A \otimes_R M^{H^*} \to M$ [Mon93, 8.3.3]. Applying this with $M = A\#H^* \otimes_A V$, where V is an A-module, yields the second diagram. Now let V be an R-module and apply the isomorphism with A replaced by $A\#H^*$, R by A and $M = A\#H^*\#H \otimes_{A\#H^*} (A \otimes_R V)$, yielding the first diagram. The remaining diagrams are clear from the definition of extended and restricted module. $\square$

Prime ideals. An ideal P of R is *H-prime* if it is H-stable and whenever I, J are H-stable ideals of R with $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

A prime ideal which is H-stable is clearly H-prime. We write $H\operatorname{Spec}(R)$ for the poset of all H-prime ideals. The equivalence relation $\sim_H$ restricts to an equivalence relation on $\operatorname{Spec}(R)$ which we again write as $\sim_H$.

2.12. Proposition. *Let (R, A) be a faithfully flat H-Galois extension.*

(i) *The maps Φ, Ψ restrict to poset isomorphisms between $H\operatorname{Spec}(R)$ and $H^*\operatorname{Spec}(A)$. We write $I \leftrightarrow J$ in this situation, and say that Φ, Ψ respect prime ideals.*

(ii) *The map $Q \mapsto (Q : H)$ is a poset map from $\operatorname{Spec}(R)$ onto $H\operatorname{Spec}(R)$, and hence induces a bijection between $\operatorname{Spec}(R)/\sim_H$ and $H\operatorname{Spec}(R)$.*

(iii) *The map $P \mapsto P \cap R$ is a poset map from $\operatorname{Spec}(A)$ onto $H\operatorname{Spec}(R)$, and hence, if H is finite-dimensional, it induces a bijection between $\operatorname{Spec}(A)/\sim_{H^*}$ and $H\operatorname{Spec}(R)$.*

$\square$

For some questions, it suffices to consider the coradical H_0 of H. Since H_0 is not necessarily a Hopf subalgebra, it is necessary to define the notion of C-stable ideal for an arbitrary subcoalgebra of H. Once this is done, as in [MSch], we can define $C\,\mathrm{Spec}(R)$ in the obvious way.

2.13. Proposition ([MSch])**.** *Let (R, A) be a faithfully flat H-Galois extension. The map $I \mapsto (I : H)$ is a poset isomorphism of $H_0\,\mathrm{Spec}(R)$ onto $H\,\mathrm{Spec}(R)$.*

2.14. Proposition. *The diagram in 2.9 yields the following commutative diagram.*

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "(:H)"'] \arrow[d, leftrightarrow] & H\,\mathrm{Spec}(R) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & H^*\,\mathrm{Spec}(A) \arrow[d, leftrightarrow] & \mathrm{Spec}(A) \arrow[l, "(:H^*)"] \arrow[d, leftrightarrow] \\
\mathrm{Spec}(A\#H^*) \arrow[r, "(:H)"] & H\,\mathrm{Spec}(A\#H^*) \arrow[r, leftrightarrow] \arrow[ur, leftrightarrow] & H^*\,\mathrm{Spec}(A\#H^*\#H) & \mathrm{Spec}(A\#H^*\#H) \arrow[l, "(:H^*)"']
\end{tikzcd}$$

The diagram in 2.10 yields the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Spec}(R') \arrow[r, "(:H')"] \arrow[d, "c"'] & H'\,\mathrm{Spec}(R') \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & {H'}^*\,\mathrm{Spec}(A') \arrow[d, leftrightarrow] & \mathrm{Spec}(A') \arrow[l, "(:{H'}^*)"'] \arrow[d, "c"] \\
\mathrm{Spec}(R) \arrow[r, "(:H)"'] & H\,\mathrm{Spec}(R) \arrow[r, leftrightarrow] & H^*\,\mathrm{Spec}(A) & \mathrm{Spec}(A) \arrow[l, "(:H^*)"]
\end{tikzcd}$$

Proof. The bijection $f : I \mapsto I'$ of 2.8 preserves products, and hence under f (and its inverse), prime ideals correspond to prime ideals. In the second diagram, contraction respects prime ideals since the extension $R \subset R'$ is centralizing. $\square$

3. Krull relations

For an extension $R \subseteq A$ of commutative rings, there is a well-behaved relationship between the prime ideals of A and those of R, given by contraction. This also holds for finite centralizing extensions. However, in general ring extensions, this correspondence no longer holds. The standard definition is that $P \in \mathrm{Spec}(A)$ lies over $Q \in \mathrm{Spec}(R)$ if and only if Q is a minimal prime over $P \cap R$.

In the faithfully flat Galois situation, there are other reasonable definitions. We can say that P lies over Q if $P \cap R = (Q : H)$. This is the definition of Montgomery and Schneider, which we shall use in what follows. Another possibility is: P lies over Q if P is minimal over $(Q : H)A$.

In the remainder of this section, we assume that H is finite-dimensional and work with a fixed faithfully flat H-Galois extension $\mathcal{E} = (R, A)$.

3.1. Definition. Let P be a prime ideal of A and Q a prime ideal of R. Say that P *lies over* Q if $(P : H^*) \leftrightarrow (Q : H)$.

The above formulation brings out the symmetry in the situation, and we may equally well say that Q lies under P. More explicitly, P lies over Q if and only if $P \cap R = (Q : H)$, if and only if $(Q : H)A = (P : H^*)$. Note that by 2.12, for each $P \in \operatorname{Spec}(A)$ there is some $Q \in \operatorname{Spec}(R)$ lying under P, and for each $Q \in \operatorname{Spec}(R)$ there is some $P \in \operatorname{Spec}(A)$ lying over Q.

It follows from 2.14 that $P \in \operatorname{Spec} A$ lies over $Q \in \operatorname{Spec}(R)$ if and only if Q' lies over P, and this occurs if and only if P' lies over Q'. Similarly, $P \in \operatorname{Spec}(A')$ lies over $Q \in \operatorname{Spec}(R')$ if and only if $P \cap A$ lies over $Q \cap R$. This enables us to streamline considerably the verification of many properties in this section, since it usually suffices to consider smash products, we can use duality systematically, and we may extend the field if necessary.

External Krull relations. Montgomery and Schneider combine the usual cutting down and lying over relations into a single stronger property which they call t-LO.

3.2. Definition. We say that $\mathcal{E}$ has *t-LO* if and only if for every $J \in H^* \operatorname{Spec}(A)$, there are $P_1, \dots, P_n \in \operatorname{Spec}(A)$ equivalent to J, such that $(\cap_i P_i)^t \subseteq J$. We say that $\mathcal{E}$ has *t-coLO* if for every $I \in H \operatorname{Spec}(R)$, there are $Q_1, \dots, Q_m \in \operatorname{Spec}(R)$ equivalent to I, such that $(\cap_i Q_i)^t \subseteq I$.

If we do not wish to specify t, then we shall just say that LO or coLO is satisfied. Note that the definition in [MSch] has the additional requirement that $m, n \leq \dim H$. We shall not require such detailed information.

3.3. Proposition. *Suppose that $\mathcal{E}$ has t-LO, let $P \in \operatorname{Spec}(A)$ and let $Q \in \operatorname{Spec}(R)$. Then*

(i) *there are at most t primes of A minimal over a given H^*-prime of A*
(ii) *if P is minimal over $J \in H^* \operatorname{Spec}(A)$ then $P \cap R = J \cap R$, and*
(iii) *if P is minimal over $(Q : H)A$ then P lies over Q.*

Suppose that $\mathcal{E}$ has t-coLO, let $P \in \operatorname{Spec}(A)$ and let $Q \in \operatorname{Spec}(R)$. Then

(i) *there are at most t primes of R minimal over a given H-prime of R*
(ii) *if Q is minimal over $I \in H \operatorname{Spec}(A)$ then $(Q : H) = I$, and*
(iii) *if Q is minimal over $P \cap R$ then P lies over Q.*

Proof. We prove only the t-coLO case as the other follows by duality. Fix an H-prime ideal I of R and let Q' be a prime of R minimal over I. By t-coLO there exist $Q_1, \dots, Q_m \in \operatorname{Spec}(R)$ such that $(Q_i : H) = I$ and $(\cap_i Q_i)^t \subseteq I \subseteq Q'$. Since Q' is prime, some $Q_i \subseteq Q'$ and by minimality $Q_i = Q'$. Hence every such Q' is one of the Q_i and $(Q' : H) = I$. This establishes (i) and (ii), and (iii) follows from the way we have defined lying over. ☐

It is natural to investigate the converse of this last implication.

3.4. Proposition. *The following conditions are equivalent for $Q \in \operatorname{Spec}(R)$:*

(i) *The prime ideals of R which are H-equivalent to Q are mutually incomparable*

(ii) *Q is minimal over $(Q : H)$*
(iii) *For $P \in \operatorname{Spec}(A)$, if P lies over Q then Q is minimal over $P \cap R$*

Proof. Assuming (i), if $(Q : H) \subseteq Q' \subseteq Q$, and $Q' \in \operatorname{Spec}(R)$ is minimal over $(Q : H)$, then $Q = Q'$, yielding (ii). If (ii) holds, then if $Q \subseteq Q'$ and $(Q : H) = (Q' : H) = I$, say, then both Q and Q' are minimal over I and hence are equal. Parts (ii) and (iii) are equivalent by definition. □

The dual result is also true.

3.5. Proposition. *The following conditions are equivalent for $P \in \operatorname{Spec}(A)$:*
(i) *The prime ideals of A which are H^*-equivalent to P are mutually incomparable*
(ii) *P is minimal over $(P : H^*)$*
(iii) *For $Q \in \operatorname{Spec}(R)$, if P lies over Q then P is minimal over $(Q : H)A$*

□

3.6. Definition. Say that $\mathcal{E}$ has *coINC* if and only if the conditions in 3.4 hold for all $Q \in \operatorname{Spec}(R)$. Say that $\mathcal{E}$ has *INC* if and only if the conditions in 3.5 hold for all $P \in \operatorname{Spec}(A)$.

Thus when coINC holds, the map $Q \mapsto (Q : H)$ is strictly increasing, and when INC holds, the map $P \mapsto P \cap R$ is strictly increasing. If coINC holds, and R is prime, then every nonzero ideal of R contains a nonzero H-stable ideal.

Few interesting results can be achieved in the presence of just one of the Krull relations. A useful combination is the property of satisfying coLO and coINC, which is transitive and holds over subfields and algebraic extension fields [MSch, Sections 6 and 7]. When this property holds, our definition of lying over is equivalent to the standard definition. This is a consequence of the next Wedderburn-type result, whose proof is routine (compare [Pas89, Theorem 16.2]).

3.7. Proposition. *H satisfies both coLO and coINC if and only if for every H-prime H-module algebra R, there are finitely many primes $Q_1, \dots, Q_m$ with $(Q_i : H) = 0$, all the Q_i are incomparable, and their intersection N is nilpotent. In this case the Q_i are precisely the minimal primes of R and N is the maximum nilpotent ideal of R.* □

We now come to going up. As noted in [MSch], going down is a consequence of LO, and so we shall omit it (but see the next subsection, where the internal version is given).

3.8. Definition. Say that $\mathcal{E}$ satisfies *going up (GU)* if whenever we have $Q_1, Q_2 \in \operatorname{Spec}(R)$ and $P_1 \in \operatorname{Spec}(A)$ such that $Q_1 \subseteq Q_2$ and P_1 lies over Q_1, then there is $P_2 \in \operatorname{Spec}(A)$ containing P_1 and lying over Q_2.

Say that $\mathcal{E}$ satisfies *co-going up (coGU)* if whenever we have $P_1, P_2 \in \operatorname{Spec}(A)$ and $Q_1 \in \operatorname{Spec}(R)$ such that $P_1 \subseteq P_2$ and P_1 lies over Q_1, then there is $Q_2 \in \operatorname{Spec}(R)$ containing Q_1 and lying under P_2.

In all cases the use of the prefix "co" is justified — it is shown in [MSch] that t-LO is dual to t-coLO, and similarly for the other properties.

Internal Krull relations. We now consider the relationship between $\operatorname{Spec}(R)$ and $H\operatorname{Spec}(R)$ in more detail, by studying "internal" versions of the lying over relations. The internal versions of t-LO, t-coLO, INC and coINC are the same as above. However for going up and down, there are differences.

3.9. Definition. Say that $\mathcal{E}$ satisfies *internal coGU* if whenever we have $I_1, I_2 \in H\operatorname{Spec}(R)$ and $Q_1 \in \operatorname{Spec}(R)$ such that $I_1 \subseteq I_2$ and $(Q_1 : H) = I_1$ then there is $Q_2 \in \operatorname{Spec}(R)$ such that $Q_1 \subseteq Q_2$ and $(Q_2 : H) = I_2$.

Say that $\mathcal{E}$ satisfies *internal GU* if whenever we have $Q_1, Q_2 \in \operatorname{Spec}(R)$ and $I_1 \in \operatorname{Spec}(R)$ such that $Q_1 \subseteq Q_2$ and $(Q_1 : H) = I_1$ then there is $I_2 \in \operatorname{Spec}(R)$ such that $I_1 \subseteq I_2$ and $(Q_2 : H) = I_2$.

3.10. Definition. Say that $\mathcal{E}$ satisfies *internal GD* if whenever we have $Q_1, Q_2 \in \operatorname{Spec}(R)$ and $I_2 \in H\operatorname{Spec}(R)$ such that $Q_1 \subseteq Q_2$ and $(Q_2 : H) = I_2$ then there is $I_1 \in H\operatorname{Spec}(R)$ contained in I_2 with $(Q_1 : H) = I_1$.

Say that $\mathcal{E}$ satisfies *internal coGD* if whenever we have $I_1, I_2 \in H\operatorname{Spec}(R)$ and $Q_2 \in \operatorname{Spec}(R)$ such that $I_1 \subseteq I_2$ and $(Q_2 : H) = I_2$, then there is $Q_1 \in \operatorname{Spec}(R)$ contained in Q_2 with $(Q_2 : H) = I_2$.

The proof of the following proposition is immediate from the definitions.

3.11. Proposition. *The following conditions hold for $\mathcal{E}$.*

(i) *The internal versions of coGU and coGD imply the corresponding external versions*
(ii) *The external versions of GU and GD imply the corresponding internal versions*

□

Note that the dual of internal coGU is not internal GU. In fact it is easy to see that this dual condition is formally stronger than GU.

3.12. Proposition. *$\mathcal{E}$ satisfies internal coGU in the following situations:*

(i) *$\mathcal{E}$ satisfies LO and coGU*
(ii) *$\mathcal{E}$ satisfies coLO, coGU and GU*
(iii) *Every prime of R is maximal and $\mathcal{E}$ satisfies either LO or coLO*

Proof. Suppose that $\mathcal{E}$ satisfies LO and coGU, and let $I_1, I_2 \in H\operatorname{Spec}(R)$ and $Q_1 \in \operatorname{Spec}(R)$ be such that $I_1 \subseteq I_2$ and $(Q_1 : H) = I_1$. Then there is $P \in \operatorname{Spec}(A)$ such that $I_2 = P \cap R$.

By LO, there exist primes $P_1, \dots, P_n$ of A such that $P_i \cap R = I_1$ and $(\cap_i P_i)^t \subseteq I_1 A \subseteq I_2 A \subseteq P$. Since P is prime, $P_i \subseteq P$ for some i. Now by coGU, there is $Q_2 \in \operatorname{Spec}(R)$ so that $Q_2 \subseteq Q_1$ and $P \cap R = (Q_2 : H)$. This proves (i).

For (ii), again let $I_1, I_2 \in H\operatorname{Spec}(R)$ and $Q_1 \in \operatorname{Spec}(R)$ be such that $I_1 \subseteq I_2$ and $(Q_1 : H) = I_1$. Write $I_1 = P \cap R, I_2 = (Q : H)$ for some $Q \in \operatorname{Spec}(R)$ and $P \in \operatorname{Spec}(A)$. By coLO, there are primes $Q_3, \dots, Q_m$ of R with $(Q_i : H) = I_1$ and $(\cap_i Q_i)^t \subseteq I_1 \subseteq I_2 \subseteq Q$. Thus some $Q_i \subseteq Q$. Now apply GU to the diagram formed by Q_i, Q and P, to obtain a prime P' of A with $P \subseteq P'$ and $P' \cap R = I_2$. Applying coGU to the diagram formed by I_1, P, P' yields $Q' \in \operatorname{Spec}(R)$ such that $(Q' : H) = P' \cap R = I_2$ and $Q \subseteq Q'$, yielding the result. Part (iii) follows from (i) and (ii) because since every prime is maximal, both GU and coGU are automatic. □

As mentioned in the previous subsection, GD is implied by LO. The dual is true, and it is easily seen that the stronger implication coLO $\Rightarrow$ internal coGD holds.

All Krull relations. The nicest situation is when all the external Krull relations hold. Note that in this case all internal relations are also satisfied. It follows in a straightforward manner from 2.14, as in [MSch], that this property is transitive and self-dual, and holds over all subfields and all algebraic extension fields. This last stipulation is because only for algebraic extensions can we be guaranteed that $R \subset R'$ has all Krull relations.

If all Krull relations hold, every strictly ascending (descending) chain in any of $H\,\mathrm{Spec}(R)$, $\mathrm{Spec}(R)$, $H^*\,\mathrm{Spec}(A)$ or $\mathrm{Spec}(A)$ yields a strictly ascending (descending) chain of the same length in each of the other posets. This enables us to compare prime heights and depths, classical Krull dimension, etc.

Group algebras of finite groups, and their duals, satisfy all Krull relations. The proof of this rests on work of Lorenz and Passman [LP79]. There are no Hopf algebras which are known not to satisfy all the Krull relations. However, for most examples, either verifying a Krull relation or else determining that it does not hold is extremely difficult.

4. Maximal ideals

As usual, in this section H is a finite-dimensional Hopf algebra and $\mathcal{E} = (R, A)$ a faithfully flat H-Galois extension.

We say that P is an H*-maximal* ideal if P is a maximal element of $\mathcal{I}_H(R)$. The poset of all such ideals is denoted by $H\,\mathrm{Max}(R)$; it is clearly contained in $H\,\mathrm{Spec}(R)$. It is immediate that an H-stable ideal which is also maximal is an H-maximal ideal. The quotient of R by an H-maximal ideal is an H*-simple* ring.

4.1. Proposition. *The maps Φ and Ψ respect maximal ideals; that is, they restrict to poset isomorphisms $H\,\mathrm{Max}\,R \leftrightarrow H^*\,\mathrm{Max}\,A$.*

Proof. The set of all H-stable ideals containing a given H-stable ideal is inductive and so by Zorn's lemma, every H-stable ideal is contained in an H-maximal ideal. Thus an H-maximal ideal is precisely a maximal element of $H\,\mathrm{Spec}(R)$. Since we know that Φ and Ψ restrict to maps between $H\,\mathrm{Spec}(R)$ and $H^*\,\mathrm{Spec}(A)$, and the set of maximal elements is a poset invariant, the result follows. $\square$

We wish to see how the lying over relations are compatible with maximal ideals.

4.2. Proposition. *The following conditions hold for $\mathcal{E}$.*

(i) *Let I be an H-maximal ideal of R. Then there is a maximal ideal M of A with $M \cap R = I$.*
(ii) *Let I be an H-maximal ideal of R. Then there is a maximal ideal M of R with $(M : H) = I$.*

Proof. Let M be a maximal ideal of A (respectively, of R) containing IA (respectively, I). $\square$

Consider the property that in 4.2(i), *every* prime ideal of A with $P \cap R = I$ is maximal. This property can be considered as a weakened form of INC, since INC implies it by 4.2. Similarly, if

coINC holds, then every prime ideal of R with core I is maximal. Thus coINC implies that every prime H-simple ring is in fact simple.

If the weakened forms of INC and coINC hold then there is no confusion when speaking of the equivalence class of a maximal ideal, as the equivalence classes in Spec and Max coincide. A priori, however, there is no reason to expect these two equivalence classes to be equal.

Now by 4.2 there is a map $H\operatorname{Max}(R) \to \operatorname{Max}(R)/\sim_H$ which is $1-1$. Similarly there is a map $H\operatorname{Max}(R) \to \operatorname{Max}(A)/\sim_{H^*}$ which is $1-1$. In contrast to the situation with Spec, it is not clear that these maps are onto. The following definitions address this question.

4.3. Definition. We say that $\mathcal{E}$ has *coMAX* if whenever M is a maximal ideal of R then $(M : H)$ is an H-maximal ideal of R. Dually, $\mathcal{E}$ has *MAX* if whenever M is a maximal ideal of A then $M \cap R$ is an H-maximal ideal of R.

It is readily seen, using 4.1, that MAX and coMAX are indeed dual to each other. Now suppose that K is a normal Hopf subalgebra of H with quotient $\overline{H}$, and consider the chain $R \subset B \subset A$ as in Section 2. Suppose that each intermediate extension has both MAX and coMAX. If M is maximal in A, then $M \cap B$ is H-maximal in B. By 4.2, there is a maximal ideal M' of B with $(M' : \overline{H}) = M \cap B$. It follows that $M \cap R = ((M' \cap R) : H)$. This latter ideal is H-maximal by MAX and coMAX applied to (R, B). Thus by 2.7, the property of satisfying MAX and coMAX is transitive.

The proof of the following proposition is immediate from the definitions.

4.4. Proposition. *If $\mathcal{E}$ satisfies internal coGU then $\mathcal{E}$ satisfies coMAX. If $\mathcal{E}^*$ satisfies internal coGU then $\mathcal{E}$ satisfies MAX.* □

4.5. Theorem. *Suppose that $\mathcal{E}$ satisfies coINC and coMAX. Then following conditions are equivalent for $Q \in \operatorname{Spec}(R)$.*

(i) *Q is maximal*
(ii) *$(Q : H)$ is H-maximal*
(iii) *Q is minimal over an H-maximal ideal of R*
(iv) *$(Q : H)A$ is H^*-maximal*

Dually, suppose that $\mathcal{E}$ satisfies INC and MAX. Then the following conditions are equivalent for $P \in \operatorname{Spec}(A)$.

(i) *P is maximal*
(ii) *$(P : H^*)$ is H^*-maximal*
(iii) *P is minimal over an H^*-maximal ideal of A*
(iv) *$P \cap R$ is H-maximal*

Proof. We prove only the equivalence of the first four conditions. Clearly (ii) and (iv) are equivalent by 4.1. Also (ii) implies (iii) by 3.4, and given (iii), the H-maximal ideal in question must be $(Q : H)$, so (ii) and (iii) are equivalent. Now (ii) implies (i) in the presence of coINC and the implication (i)⇒(ii) is precisely coMAX. □

4.6. Corollary. *Consider the property of H: if $P \in \operatorname{Spec}(A)$ lies over $Q \in \operatorname{Spec}(R)$, then P is maximal if and only if Q is maximal.*

This property is self-dual, transitive and field-independent, and it holds if H satisfies MAX, coMAX, INC and coINC. □

The property in 4.6 can be considered as a minimal requirement for a decent theory of lying over. Recall that as far as we know, every finite-dimensional H satisfies the hypotheses of 4.6.

5. Modules

Let $\mathcal{E} = (R, A)$ be a ring extension, V an R-module, and W an A-module. Then we denote the restricted R-module $W_{|R}$ by $W^{\downarrow}$ and the induced A-module $A \otimes_R V$ by $V^{\uparrow}$.

5.1. Proposition. *Suppose that $\mathcal{E}$ is faithfully flat H-Galois, and let V be an R-module and W an A-module. Then the following conditions hold.*

(i) $\operatorname{ann} W^{\downarrow} = (\operatorname{ann} W) \cap R$
(ii) $\operatorname{ann} V^{\uparrow\downarrow} = (\operatorname{ann} V : H)$
(iii) *If H is finite-dimensional, then* $\operatorname{ann} V^{\uparrow} = (\operatorname{ann} V : H)A$
(iv) *If H is finite-dimensional, then* $\operatorname{ann} W^{\downarrow\uparrow} = (\operatorname{ann} W : H^*)$
(v) *If V is simple and X is a simple image of $V^{\uparrow}$, then* $\operatorname{ann} X$ *lies over* $\operatorname{ann} V$
(vi) *If W is simple and X is a simple image of $W^{\downarrow}$, then* $\operatorname{ann} W$ *lies over* $\operatorname{ann} X$

Proof. Part (i) is trivial. For (ii), let $Q = \operatorname{ann} V, P = \operatorname{ann} V^{\uparrow}$. Then $P \cap R \subseteq Q$ since $V^{\uparrow\downarrow}$ has an R-submodule isomorphic to V. Since $P \cap R$ is the intersection of an ideal of A with R, it is H-stable and so $P \cap R \subseteq (Q : H)$. But clearly $(Q : H)AV = A(Q : H)V = 0$ so that $(Q : H) \subseteq P \cap R$. This yields (ii). Parts (iii) and (iv) follow from the observation that $\operatorname{ann} V^{\uparrow}$, as the annihilator of an H^*-stable module, is H^*-stable.

We now prove (v). Since X is simple, $X = V^{\uparrow}/M$ for some maximal A-submodule M of $V^{\uparrow}$. Let $P = \operatorname{ann} X, Q = \operatorname{ann} V$. Then $(Q : H)A \subseteq P$ by (ii). Also $(P \cap R)V$ is an R-submodule of V. Since $(P \cap R)V^{\uparrow} \subseteq M \neq V^{\uparrow}$ and $P \cap R$ commutes with A it follows that $(P \cap R)V \neq V$ and so $(P \cap R)V = 0$, since V is a simple R-module. Thus $P \cap R \subseteq Q$ and since $P \cap R$ is H-stable, (v) follows.

Part (vi) follows in a similar manner. If W is simple, let $P = \operatorname{ann} W$, write $X = W^{\downarrow}/M$ for some maximal R-submodule M of $W^{\downarrow}$, and let $Q = \operatorname{ann} V$. Clearly $P \cap R \subseteq Q$ and so $P \cap R \subseteq (Q : H)$. Conversely $(Q : H)W$ is an A-submodule of W and so since $(Q : H)W \subseteq M \neq W$, we have $(Q : H)W = 0$. Thus the reverse containment is shown and we obtain $(Q : H) = P \cap R$. □

5.2. Definition. Say that $\mathcal{E} = (R, A)$ has the *finite induction property* if whenever V is an R-module of finite length, then $V^{\uparrow}$ is an A-module of finite length.

It is easy to see that H has the finite induction property if and only if H^* has the dual *finite restriction property*, that the restriction of each finite length A-module to R is of finite length.

Clearly, both the finite induction and finite restriction properties are transitive. It follows from 2.11 that they are field-independent. They are both satisfied by group algebras, and the finite

induction property by restricted enveloping algebras [Chi87, Lemma 23] (we reproduce a proof of a generalization of this last result in section 9). In fact these Hopf algebras have the stronger property that if V has finite length, then so does $V^{\uparrow\downarrow}$.

5.3. Definition. Say that $\mathcal{E} = (R, A)$ has the *semisimple induction property* if whenever V is semisimple of finite length then so is $V^{\uparrow}$. Dually, $\mathcal{E}$ has the *semisimple restriction property* if whenever W is semisimple of finite length then so is $W^{\downarrow}$.

Again, the semisimple induction and restriction properties are transitive and field-independent. The latter property is always satisfied by group algebras, as is the former in characteristic coprime to the group order. The semisimple restriction property is in fact satisfied by finite normalizing extensions (see, for example, [Pas81]). In addition, such extensions satisfy the property that $V^{\uparrow\downarrow}$ is semisimple of finite length whenever V is.

6. Primitive ideals

In this section, H is finite-dimensional and $\mathcal{E} = (R, A)$ a fixed faithfully flat Galois H-extension.

It is natural to define the set $H\operatorname{Prim}(R)$ of *H-primitive* ideals. Unlike the case for prime and maximal ideals, there does not seem to be any definition which is internal to R. In the case where $A = R\#H$ is a smash product, the obvious definition is that Q is H-primitive if and only if Q is the annihilator of an H-stable module. The only reasonable definition of H-stable module for R is a module for $R\#H$. Thus Q should be H-primitive if and only if $Q = P \cap R$ for some $P \in \operatorname{Prim}(A)$.

However, in more general extensions, this notion of H-stable module does not make sense, since there need be no action of H on R. Any definition of H-primitive should satisfy the following requirements. First, we should have $H\operatorname{Max} R \subseteq H\operatorname{Prim}(R) \subseteq H\operatorname{Spec}(R)$. Second, an H-stable ideal which is primitive should be H-primitive. Third, the bijections between $H\operatorname{Spec}(R)$ and $H^*\operatorname{Spec}(A)$ should yield bijections between $H\operatorname{Prim}(R)$ and $H^*\operatorname{Prim}(A)$.

The definition we choose yields the desired results fairly quickly, and in order to obtain a primitive analogue of 2.14, it is the obvious one. However, some problems remain. In order to obtain an analogue of 2.13, it would be necessary to define C-primitive ideals for an arbitrary subcoalgebra of H. No obvious candidate presents itself.

6.1. Definition. An ideal I of R is *H-primitive* if $I = P \cap R$ for some primitive ideal P of A.

Clearly every H-primitive ideal is H-prime. Let I be an H-maximal ideal of R. Then by 4.2, $I = M \cap R$ for some maximal ideal M of A, so that I is H-primitive. Thus the first requirement of our definition is satisfied. The second and third follow from the next result, which shows that another candidate for the definition of H-primitive is equivalent to the one given. Thus, the intersections with R of primitive ideals of A coincide with the H-cores of primitive ideals of R.

6.2. Proposition. *The following conditions hold for $\mathcal{E}$.*

(i) *The map $Q \mapsto (Q : H)$ is a poset epimorphism from* $\operatorname{Prim}(R)$ *onto* $H\operatorname{Prim}(R)$*. This induces a bijection between* $\operatorname{Prim}(R)/\sim_H$ *and* $H\operatorname{Prim}(R)$.

(ii) *The map $P \mapsto P \cap R$ is a poset epimorphism from* $\operatorname{Prim}(A)$ *onto* $H\operatorname{Prim}(R)$*. This induces a bijection between* $\operatorname{Prim}(A)/\sim_{H^*}$ *and* $H\operatorname{Prim}(R)$.

(iii) *The poset isomorphism* $H\operatorname{Spec}(R) \leftrightarrow H^*\operatorname{Spec}(A)$ *respects primitive ideals; that is, it restricts to a poset isomorphism* $H\operatorname{Prim}(R) \leftrightarrow H^*\operatorname{Prim}(A)$*.*

Proof. Part (ii) is immediate from the way we have defined H-primitive ideals. We prove (i). We first show that the map is well defined. Let $Q \in \operatorname{Prim}(R)$. Then $Q = \operatorname{ann} V$ for some simple module V. Then $V^{\uparrow}$ is finitely generated, hence has a maximal submodule and therefore a simple quotient. Applying 5.1(v) we obtain $P \in \operatorname{Prim}(A)$ such that $P \cap R = (Q : H)$. We now show that the map is onto. Given $I \in H\operatorname{Prim}(R)$, write $I = P \cap R$ where $P \in \operatorname{Prim}(A)$. Then $P = \operatorname{ann} W$ for some simple A-module W. Since A is a finitely generated R-module, $W^{\downarrow}$ is finitely generated. Similarly to the above, applying 5.1(vi) we obtain $Q \in \operatorname{Prim}(R)$ with $(Q : H) = I$. Thus the map is onto, proving (i). Part (iii) is now immediate. □

In particular, this last result shows that the analogue for primitive ideals of coMAX is always satisfied (the analogue of MAX is satisfied by definition).

Since the ideal equivalence of 2.8 is a composition of two equivalences of (bi)module categories and hence preserves exact sequences, it follows that under this map, primitive ideals correspond to primitive ideals. This, and the fact that primitive ideals behave well with respect to lying over in centralizing extensions, shows that the analogue for Prim of 2.14 holds, if we restrict all maps from Spec to Prim.

It is clear that each of the Krull relations of section 3 can be formulated for Prim, so that we can speak of, for example, t-LO for primitive ideals. Obviously INC and coINC for Spec imply the corresponding relations for Prim, but the situation with regard to the other relations is not immediately apparent. However, a construction of Passman, the "primitivity machine" [Pas81], can be adapted to our situation, as we shall now explain.

For every ring R, there is an overring $S = \widehat{R} = R\langle X\rangle$ which is obtained from R by taking (noncommutative) formal power series and polynomials. The variables adjoined all commute with R. Key properties are: $\widehat{I} \cap R = I, \widehat{R}/\widehat{I} \cong \widehat{R/I}, \widehat{A}\widehat{B} \subseteq \widehat{AB}, \widehat{\cap_i A_i} = \cap_i \widehat{A_i}$. Here I is an ideal of R and A a subset of R. If Q is a prime ideal of R then $\widehat{Q}$ is a primitive ideal of $\widehat{R}$. Furthermore, the map $\widehat{\ }$ is injective on ideals and if I is an annihilator ideal of $\widehat{R}$, then $I \cap R$ is a prime ideal of R.

If H acts on R, then the action extends naturally to an action on S, by letting H act trivially on the adjoined variables in X. Furthermore it is clear that if $A = R\#H$ then $\widehat{A} = \widehat{R}\#H$. We summarize the main facts on spectra in the following result, whose proof follows directly from 2.12, 6.2 and the remarks above.

6.3. Theorem. *The following diagram commutes. The maps* ι *are the natural inclusions, and the vertical maps come from 2.12 and 6.2.*

$$\begin{tikzcd}
\mathrm{Prim}(A) \arrow[r, "\iota"] \arrow[d, "(:H^*)"'] & \mathrm{Spec}(A) \arrow[r, "\widehat{\ }"] \arrow[d, "(:H^*)"'] & \mathrm{Prim}(\widehat{A}) \arrow[d, "(:H^*)"] \\
H^*\,\mathrm{Prim}(A) \arrow[r, "\iota"] \arrow[d, leftrightarrow] & H^*\,\mathrm{Spec}(A) \arrow[r, "\widehat{\ }"] \arrow[d, leftrightarrow] & H^*\,\mathrm{Prim}(\widehat{A}) \arrow[d, leftrightarrow] \\
H\,\mathrm{Prim}(R) \arrow[r, "\iota"] & H\,\mathrm{Spec}(R) \arrow[r, "\widehat{\ }"] & H\,\mathrm{Prim}(\widehat{R}) \\
\mathrm{Prim}(R) \arrow[r, "\iota"] \arrow[u, "(:H)"] & \mathrm{Spec}(R) \arrow[r, "\widehat{\ }"] \arrow[u, "(:H)"] & \mathrm{Prim}(\widehat{R}) \arrow[u, "(:H)"']
\end{tikzcd}$$

□

It is a consequence of 6.3 that each Krull relation for Prim implies the analogous one for Spec. We give one example here. Suppose that H satisfies INC for Prim. Let $P_1, P_2 \in \mathrm{Spec}(A)$ with $P_1 \subset P_2$. Then $\widehat{P_1} \subset \widehat{P_2}$, and so since INC holds for Prim, we have $\widehat{P_1} \cap \widehat{R} \subset \widehat{P_2} \cap \widehat{R}$. Thus we must have $P_1 \cap R \subset P_2 \cap R$ and so INC holds for Spec. See the discussion of cocommutative Hopf algebras in section 9, or [Pas81] for more details.

We now investigate the converse implications.

6.4. Proposition. *If $\mathcal{E}$ satisfies the finite induction property (respectively, the finite restriction property), then it satisfies LO (respectively coLO) for primitive ideals.*

Proof. We prove only the second assertion as the first then follows by duality. Let P be a primitive ideal of A, with $P = \mathrm{ann}\, W$. By 5.1, $\mathrm{ann}\, W^{\downarrow} = P \cap R$, an H-primitive ideal of R. Let $Q_1, \dots, Q_m$ be the annihilators of the finitely many composition factors of $W^{\downarrow}$, where $Q_i, 1 \leq i \leq s \leq m$, are the distinct ones. Then whenever t is such that $st \geq m$, $(\cap_i Q_i)^t \subseteq \mathrm{ann}\, W^{\downarrow} = P \cap R$ as required. □

The proof of the following proposition is essentially the same as the argument of 3.3.

6.5. Proposition. *The following conditions hold for $\mathcal{E}$.*

(i) *Suppose that $\mathcal{E}$ has LO for primitive ideals. If $Q \in \mathrm{Prim}(R)$ and $P \in \mathrm{Spec}(A)$ is minimal over $(Q : H)A$ then P is primitive.*

(ii) *Suppose that $\mathcal{E}$ has coLO for primitive ideals. If $P \in \mathrm{Prim}(A)$ and $Q \in \mathrm{Spec}(R)$ is minimal over $P \cap R$ then Q is primitive.* □

6.6. Theorem. *If $\mathcal{E}$ has coINC and coLO for primitive ideals, then the following conditions are equivalent for $Q \in \mathrm{Spec}(R)$.*

(i) *Q is primitive*
(ii) *$(Q : H)$ is H-primitive*
(iii) *Q is a minimal prime over an H-primitive ideal of R*
(iv) *$(Q : H)A$ is H^*-primitive*

If $\mathcal{E}$ has INC and LO for primitive ideals, then the following conditions are equivalent for $P \in \mathrm{Spec}(A)$.

(i) *P is primitive*
(ii) *$(P : H^*)$ is H-primitive*
(iii) *P is a minimal prime over an H^*-primitive ideal of A*
(iv) *$P \cap R$ is H-primitive*

Proof. We prove only the equivalence of the first four conditions, the others following by duality. The implication (i)⇒ (ii) is 6.2(i), and (ii)⇒ (iii) follows from coINC. Also (ii) and (iv) are equivalent by 6.2(iii). Now (iii) implies (i) by 6.5 and 6.2(ii). □

6.7. Theorem. *Consider the following property of H: if $P \in \mathrm{Spec}(A)$ lies over $Q \in \mathrm{Spec}(R)$, then P is primitive if and only if Q is primitive ("lying over respects primitivity").*

This property is self-dual, transitive, and field-independent, and it holds if (i) H satisfies INC, coINC, LO and coLO for primitive ideals or (ii) H and H^* are pointed. Conversely, if lying over respects primitivity and H has all Krull relations for Spec*, then H has all Krull relations for* Prim.

Proof. The property is self-dual, transitive and field-independent by 2.14 and the remarks after 6.2. It holds in case (i) by 6.6. In case (ii), we use the fact (see Section 9) that for every pointed H, the primes minimal over a given H-prime ideal of R are all conjugate under $G(H)$, so that if one of them is primitive, then they all are. By 6.2 there is one such primitive ideal. The case where H^* is pointed follows by duality. If H has all Krull relations for Spec and lying over respects primitivity, then if we start with primitive ideals, all prime ideals produced by LO, GU and their duals are primitive, as required. We already know that INC and coINC are inherited by Prim from Spec. □

As with 4.6, the property discussed in 6.7 can be considered a minimum requirement of a decent theory of lying over. It holds for finite normalizing extensions [MR87], and the extensions studied by E. Letzter in [Let89], namely ring extensions $R \subseteq S$ where R and S are noetherian and S has finite GK-dimension over R on the left and the right.

Suppose that all Krull relations for Prim are satisfied (so that also all Krull relations for Spec hold, and lying over respects primitivity). This enables us to prove many results relating Spec and Prim. For example, A satisfies the property that every primitive ideal is maximal if and only if R does. The result was first established for group algebras in [Lor78, Theorem 1.7].

7. Extensions with a total integral

The faithfully flat Galois property does not always hold for H-extensions which arise in practice, and can be difficult to verify. As was the case in [MSch], some of our results will extend to a larger class of H-extensions, namely those with a total integral. We shall not go into great detail, but confine our discussion to the properties of such extensions which enable a reasonably systematic translation of results on faithfully flat Galois extensions to the more general context.

The key facts about such an H-extension (R, A) are as follows. See [MSch] for more details. In (iv), the lying over relation is the standard one.

7.1. Proposition. *Suppose that H is finite-dimensional, and that (R, A) is an H-extension with a total integral. Then the following statements hold.*

(i) *There is an idempotent $e \in S = A\#H^*$ such that $eSe = Re \cong R$.*
(ii) *This induces a lattice map $\mathcal{I}(S) \to \mathcal{I}(R)$ which in turn induces poset epimorphisms* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *and* $\operatorname{Prim}(S) \to \operatorname{Prim}(R)$.
(iii) *These latter maps are such that $ePe = R$ if and only if $e \in P$, and they yield poset isomorphisms* $\operatorname{Spec}_e(S) \equiv \{P \in \operatorname{Spec}(S) | e \notin P\} \to \operatorname{Spec}(R)$ *and* $\operatorname{Prim}_e(S) \to \operatorname{Prim}(R)$.
(iv) *If H has coLO and coINC, then P lies over eQe if and only if Q lies over P.*

$\square$

If H is finite-dimensional and cosemisimple, then every H-extension has a total integral (see [Mon93, section 4.3]). On the other hand, if H is connected then an extension with a total integral is necessarily a crossed product extension [Bel].

It follows in a straightforward manner from 7.1 (see [MSch, Section 5]) that if H has all Krull relations for Spec, then (R, A) inherits the analogues of LO, coLO, INC and coINC. Similar results can be obtained for Prim and the details are left to the reader. However, it is known that not all Krull relations hold for such extensions; an example of Montgomery and Small [MS84] shows that the analogue of coGU fails even for the group algebra of a group of order 2.

We give one simple example of how 7.1 can be applied. Let R be the universal enveloping algebra of a finite-dimensional nilpotent Lie algebra over a field F of characteristic zero, and let G be a finite group of automorphisms of R. Then it is well known that every primitive ideal of R is maximal. Hence the same is true for $R\#G$. We claim that the same is also true of R^G. The extension $R^G \subseteq R$ has a total integral since FG is semisimple by Maschke's theorem. Let P be a primitive ideal of R^G. Then $P = eP'e$ for some primitive ideal of $R\#G$. But then P' is maximal and thus so is P.

8. Strongly semiprimitive Hopf algebras

We begin by defining the various obvious notions of radical. As usual, H is a Hopf algebra which is not assumed to be finite-dimensional. We denote the prime radical by $N(R)$ and the Jacobson radical by $J(R)$.

8.1. Definition. Let (R, A) be a faithfully flat H-extension. The *H-prime radical* $N_H(R)$ of R is the intersection of all H-prime ideals of R. The *H-Jacobson radical* $J_H(R)$ of R is the intersection of all H-primitive ideals of R.

8.2. Proposition. *The following conditions hold for a faithfully flat H-Galois extension (R, A).*

(i) $J_H(R) = J(A) \cap R = (J(R) : H)$
(ii) $N_H(R) = N(A) \cap R = (N(R) : H)$

Proof. In (i), the first equality follows immediately from the definition of H-primitive, and the second from 6.2. In (ii), both parts follow from 2.12. □

Because $J_H(R) = (J(R) : H)$, the fact that J_H really is a radical in the usual sense (for, say, the class of all H-module algebras) follows readily from the fact that J is a radical. A complete treatment would characterize J_H in terms of maximal H-stable left (or right) ideals and left (or right) quasiregularity, and N_H in terms of an H-stable analogue of the transfinite Baer process for $N(R)$. We shall not pursue this here as our main interest is in algebras with zero radical.

The following proposition summarizes, in our notation, several results in [MSch, Section 8]. We say that two conditions are equivalent on a class of extensions if whenever one condition holds for all extensions in the class, then the other hold for all extensions. It is not necessarily the case that both conditions are equivalent for a fixed extension in the class.

8.3. Proposition. *The following conditions are equivalent on the class of all faithfully flat H-Galois extensions (R, A).*

(i) *If R is H-semiprime then it is semiprime*
(ii) *$N(R)$ is H-stable*
(iii) $N_H(R) = N(R)$
(iv) $N(R) \subseteq N(A)$
(v) *If A is semiprime then R is semiprime*

Suppose now that H is finite-dimensional and that these conditions do hold for all faithfully flat H-Galois extensions. Then (iv) and (v) in fact hold for all H-extensions. □

In the situation of the last paragraph of 8.3, then we say that H is *strongly cosemiprime.* When H is finite-dimensional, the dual property, strong semiprimeness, is also of interest. H is strongly semiprime if and only if whenever R is an H-semiprime H-module algebra, then $R\#H$ is semiprime. This is of course equivalent to many other conditions dual to those in 8.3. Both properties are transitive and field-independent.

We move on to primitive ideals, with the aim of obtaining analogous results.

8.4. Proposition. *The following conditions are equivalent on the class of all faithfully flat H-Galois extensions (R, A).*

(i) *If R is H-semiprimitive then it is semiprimitive*
(ii) *$J(R)$ is H-stable*
(iii) $J_H(R) = J(R)$
(iv) $J(R) \subseteq J(A)$
(v) *If A is semiprimitive then R is semiprimitive*

Suppose that H is finite-dimensional and that all conditions hold for all faithfully flat H-Galois extensions. Then (iv) and (v) in fact hold for all H-extensions.

Proof. Clearly (v) and (i) are equivalent by 6.2. Also (ii), (iii) and (iv) are equivalent for a given extension by 8.2. The remaining equivalences are obtained by reducing to the faithfully flat H-Galois extension $(R/J), A/AJ)$, where $J = J_H(R)$.

Suppose that all the conditions hold for every faithfully flat H-Galois extension. It suffices to prove that (v) holds for all extensions. Applying (v) in the case $H^* \subseteq H^* \# H$, we see that H is cosemisimple and so every H-extension has a total integral. If now A is semiprimitive then so is $S = A \# H^*$. Thus there are primitive ideals P_i of S with $\bigcap_i P_i = 0$. It follows from 7.1 that eP_ie is either R or a primitive ideal of R. Thus deleting those which equal R we obtain a set of primitive ideals of R with intersection $e0e = 0$, and so R is semiprimitive. □

A Hopf algebra H satisfying (i)-(v) above for all faithfully flat Galois H-extensions shall be called *strongly cosemiprimitive*. The characterization of the dual property, strong semiprimitivity, is left to the reader. Both properties are transitive and field-independent.

Group algebras are certainly strongly cosemiprimitive (since the Jacobson radical is a characteristic ideal), and whenever they are semisimple (that is, in characteristic coprime to the group order), they are strongly semiprimitive [Vil58].

We now relate strong semiprimitivity to other important properties.

8.5. Theorem. *Let H be a finite-dimensional Hopf algebra and let $\mathcal{E} = (R, A)$ be a faithfully flat H-Galois extension.*

(i) *If $\mathcal{E}$ satisfies the semisimple induction property then it satisfies 1-LO for primitive ideals.*
(ii) *If H satisfies 1-LO for primitive ideals then H is strongly semiprimitive.*
(iii) *If H is strongly (co)semiprimitive then H is strongly (co)semiprime.*

Proof. Let J be an H^*-primitive ideal of A. By 6.2, $J = (Q : H)A$ where Q is a primitive ideal of R. Let V be a simple R-module with annihilator Q. Then $V^{\uparrow}$ is semisimple of finite length and by 5.1(iii) its annihilator is J. Let $P_1, \dots, P_n$ be the annihilators of the composition factors of $V^{\uparrow}$. By 5.1(v), each P_i lies over Q. Also $\cap_i P_i$ annihilates $V^{\uparrow}$ and so this intersection is contained in J. This proves (i). Now (ii) follows by taking $J = 0$ and using the conclusion of (i). For (iii) it suffices by duality to prove only that strongly semiprimitive implies strongly semiprime. For this, we use the primitivity machine. If $A = R \# H$ is semiprime then $\widehat{A}$ is semiprimitive, so that $\widehat{R}$ is semiprimitive and hence R is semiprime, yielding the desired result. □

9. Special classes of Hopf algebras

Pointed Hopf algebras. Let H be a finite-dimensional pointed Hopf algebra, with coradical filtration of length t and $\dim H_0 = d$. Then it is known [MSch, 4.10] that H satisfies coINC, t-coLO, GU and coGU. It follows from 3.12 that H and H^* satisfy internal coGU, and hence both MAX and coMAX by 4.4.

We now show that H satisfies the finite induction property. In fact this follows from [Sch90b, 1.5, 2.1], and I thank Schneider for pointing this out to me. Those results state that for every simple R-module V, the module $V^{\uparrow\downarrow}$ has a filtration $0 = V_0 \subset V_1 \cdots \subset V_n = V$ with finitely many terms. Furthermore each quotient is a finite direct sum of modules V_g, where $g \in G(H)$. These V_g are

all simple R-modules and so $V^{\uparrow\downarrow}$ has finite length. Thus so does $V^{\uparrow}$. In fact, the length of $V^{\uparrow\downarrow}$ is bounded by some function depending only on d and t in this situation.

It now follows from 6.4 that H has s-LO for Prim, for some s depending only on t and d. Using 6.3 we can show the same for Spec. Let $Q \in \mathrm{Spec}(R)$ and let $I = (Q : H)$. Then $\widehat{Q} \in \mathrm{Prim}(\widehat{R})$. Thus there exist $P_1, \ldots, P_n \in \mathrm{Prim}(\widehat{A})$ such that for each i, $P_i \cap \widehat{R} = (\widehat{Q} : H)$ and $(\cap_i P_i)^t \subseteq (\widehat{Q} : H)\widehat{A}$. Let $P_i' = P_i \cap A$. Then $P_i' \in \mathrm{Spec}(A)$ and we have $P_i' \cap R = P_i \cap \widehat{R} \cap R = (\widehat{Q} : H) \cap R = \widehat{I} \cap R = I$, and $(\cap_i P_i')^t \subseteq (\cap_i P_i)^t \cap A \subseteq (\widehat{Q} : H)\widehat{A} \cap A \subseteq \widehat{I}\widehat{A} \cap A \subseteq \widehat{IA} \cap A = IA$. Thus H has s-LO for Spec. This answers an open problem from [MSch].

Since H also has coINC, GU, coGU and coLO for Spec by the above, in order to show that all Krull relations hold it will be necessary only to prove INC.

Now suppose that R is an H-prime H-module algebra. Then combining the above with 2.13 we have

- $Q \in \mathrm{Spec}(R)$ is minimal if and only if $(Q : H) = 0$,
- the minimal primes of R are all conjugate under $G(H)$, there are at most d of them, and their intersection is nilpotent of index at most $t + 1$, and
- there is a bijection between $H\,\mathrm{Spec}(R)$ and the space of orbits of $G(H)$ on $\mathrm{Spec}(R)$.

More generally, we may obtain all the results above for *virtually pointed* H (that is, those which become pointed after a finite field extension). This includes all cocommutative H.

If both H and H^* are (virtually) pointed then H satisfies all Krull relations for both Prim and Spec, as well as the finite induction and restriction properties.

Connected Hopf algebras. Suppose that H is connected. Then since coINC is satisfied, it follows from 2.13 that the map $Q \mapsto (Q : H)$ is a poset isomorphism between $\mathrm{Spec}(R)$ and $H\,\mathrm{Spec}(R)$. Furthermore this map respects primitive and maximal ideals. By the pointed case above, if R is semiprime and H-prime then R is in fact prime. This generalizes [BMP87, Lemma 1.2(ii)]. Dually, if H^* is connected then $P \mapsto P \cap R$ is a poset isomorphism between $\mathrm{Spec}(A)$ and $H\,\mathrm{Spec}(R)$. If also the extension is centralizing then we obtain an isomorphism between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(R)$. In particular, if A is semiprime then so is R. Note that though this map is given by contraction, its inverse is not given by expansion. Indeed, 0 is not a prime ideal for any nontrivial Hopf algebra, yet the extension $F \subseteq H$ is centralizing. Applying the correspondence above to this latter extension recovers the easy fact that H must be a local ring with unique maximal ideal its augmentation ideal.

If H and H^* are both connected then again $\mathrm{Spec}(R)$ and $\mathrm{Spec}(A)$ are isomorphic, and all Krull relations and the finite induction and restriction properties are satisfied. We present a simple example of such an H. Let H be the truncated divided power Hopf algebra in one variable over F. As an algebra, H is spanned by elements $x^{(i)}$ for $0 \leq i \leq p-1$. Here $x^{(0)} = 1$ and the multiplication is given by $x^{(i)}x^{(j)} = \binom{i+j}{j}x^{(i+j)}$. Thus as an algebra, $H \cong F\langle t \mid t^p = 0\rangle$. The comultiplication is given by making $t = x^{(1)}$ primitive, the augmentation ideal is precisely (t) and the antipode is given by $S(t) = -t$.

Now H is connected and self-dual. A comodule algebra A for H is then just an algebra equipped with a derivation d which is nilpotent of index p. The coinvariants are the constants for d. The extension is faithfully flat Galois if and only if there is an element $a \in A$ with $d^{p-1}(a) = 1$.

Semisimple Hopf algebras. Montgomery and Witherspoon [MW] introduced the concept of *semisolvable* Hopf algebra, namely one with a subnormal series all of whose factors are commutative or cocommutative. Classical structure theorems show that a semisimple commutative Hopf algebra, after an algebraic extension E of the ground field F, has the form $(EG)^*$. In characteristic zero, after such an extension, a cocommutative semisimple Hopf algebra has the form EG (in characteristic zero), while in characteristic $p > 0$ it must be of the form $(EL)^* \# EG$, where L is a p-group. In each case, the factors satisfy all Krull relations, as well as the finite induction and restriction properties. In characteristic zero, the semisimple induction and restriction properties also hold for the factors. It now follows that every semisimple semisolvable Hopf algebra in characteristic zero is strongly semiprimitive and strongly cosemiprimitive, and has all Krull relations.

All known semisimple Hopf algebras of dimension less than 60, as well as many other examples, are semisolvable [Mon]. Thus the most pressing test question is whether the Hopf algebras constructed by Nikshych in [Nik97], as deformations of the group algebra of the alternating group A_5, satisfy INC and the semisimple induction property.

10. Conclusion

In this paper it has been shown that the Krull relations for Prim imply those for Spec, and that strongly semiprimitive Hopf algebras are strongly semiprime. Thus in some sense it is superfluous to consider Spec and the whole theory can be founded on Prim. Since strong semiprimitivity is implied by an appealing module property, it seems clear that the behaviour of simple modules in faithfully flat Galois extensions should be a priority for further work.

Many conjectures are suggested by the previous sections. Although many of them are obvious and it is highly unclear how to attack them, nevertheless we shall list some explicitly (as questions) in the hope of stimulating further work.

Let H be a finite-dimensional Hopf algebra.

(i) Does H satisfy all Krull relations for Prim?
(ii) Does H satisfy all Krull relations for Spec?
(iii) Does H satisfy the finite induction (restriction) property?

The answer to none of these questions is known, even for semisimple, pointed, or connected Hopf algebras. Apart from group algebras and their duals, there is no "naturally occurring" class of Hopf algebras for which all Krull relations are known to be satisfied. As noted above, the missing relation for virtually pointed H is INC. In the cocommutative case each Krull relation can be reduced (as in Section 9) to the special case of restricted enveloping algebras $u(L)$. However, all Krull relations hold when L is solvable (by transitivity and the results for abelian L obtained by Chin [Chi87]). Thus a good test problem is to try to show that INC holds for $u(sl_2)$.

We conclude with two problems on semisimple H.

(iv) Does every semisimple H satisfy the semisimple induction property?

(v) Is every semisimple H strongly semiprimitive?

I thank Susan Montgomery for her hospitality at the University of Southern California where this work was started, and her and Hans-Jürgen Schneider for helpful discussions, particularly on the proof of 3.12. I also thank the Department of Mathematics, Statistics and Computer Science at the University of Illinois at Chicago for their hospitality during the writing of this paper.

References

[Bel] A. D. Bell, *Comodule algebras and Galois extensions relative to polynomial algebras, free algebras, and enveloping algebras*, preprint (1997).

[BMP87] J. Bergen, S. Montgomery, and D. S. Passman, *Radicals of crossed products of enveloping algebras*, Israel J. Math. **59** (1987), 167–184.

[Chi87] W. Chin, *Prime ideals in restricted differential operator rings*, Israel J. Math. **60** (1987), 236–256.

[Let89] E. Letzter, *Primitive ideals in finite extensions of Noetherian rings*, J. London Math. Soc. (2) **39** (1989), 427–435.

[Lor78] M. Lorenz, *Primitive ideals in crossed products and rings with finite group actions*, Math. Z. **158** (1978), 285–294.

[LP79] M. Lorenz and D. S. Passman, *Prime ideals in crossed products of finite groups*, Israel J. Math. **33** (1979), 89–132.

[Mon93] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, 1993.

[Mon] S. Montgomery, *Classifying finite-dimensional semisimple Hopf algebras*, preprint.

[MR87] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd., Chichester, 1987.

[MS84] S. Montgomery and L. W. Small, *Integrality and prime ideals in fixed rings of P.I. rings*, J. Pure Appl. Algebra **31** (1984), 185–190.

[MSch] S. Montgomery and H.-J. Schneider, *Prime ideals in Hopf Galois extensions*, Israel J. Math. (to appear).

[MW] S. Montgomery and S. J. Witherspoon, *Irreducible representations of crossed products*, J. Pure Appl. Algebra (to appear).

[Nik97] D. Nikshych, *K_0-rings and twisting of finite-dimensional semisimple Hopf algebras*, Comm. Algebra **26**, (1998), 321–342.

[Pas81] D. S. Passman, *Prime ideals in normalizing extensions*, J. Algebra **73** (1981), 556–572.

[Pas89] D. S. Passman, *Infinite crossed products*, Pure and Applied Mathematics, vol. 135, Academic Press Inc., Boston, MA, 1989.

[Sch90a] H.-J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel J. Math. **72** (1990), 167–195.

[Sch90b] H.-J. Schneider, *Representation theory of Hopf Galois extensions*, Israel J. Math. **72** (1990), no. 1-2, 196–231.

[Sch92] H.-J. Schneider, *Normal basis and transitivity of crossed products for Hopf algebras*, J. Algebra **152** (1992), no. 2, 289–312.

[Vil58] O. E. Villamayor, *On the semisimplicity of group algebras*, Proc. Amer. Math. Soc. **9** (1958), 621–627.

Department of Mathematics, University of Auckland, Private Bag 92019 Auckland, New Zealand
E-mail address: `wilson@math.auckland.ac.nz`